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Invariance principles for sums of extreme sequential order statistics attracted to Lévy processes

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Abstract

The paper establishes strong convergence results for the joint convergence of sequential order statistics. There exists an explicit construction such that almost sure convergence to extremal processes follows. If a partial sum of rowwise i.i.d. random variables is attracted by a non-Gaussian limit law then the results apply to invariance principles for sums of extreme sequential order statistics which turn out to be almost surely convergent or convergent in probability in $D[0,1]$. Under certain conditions they converge to the non-Gaussian part of the Lévy process. In addition, we get an approximation of these Lévy processes by a finite number of extremal processes.

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1. Introduction

Let $X_{n,i}$, $1 \leqslant i \leqslant n$, denote a rowwise i.i.d. triangular array of real random variables with convergent partial sums. Then it is well-known that the sequential partial sums

$$\sum_{i=1}^{[nt]} X_{n,i} \to Z_t, \quad 0 \leqslant t \leqslant 1, \tag{1.1}$$

are convergent in distribution in the Skorohod space $D[0,1]$, see Gihman and Skorohod (1979, Chapter II, p. 199). The process $(Z_t)_{t\in[0,1]}$ is a Lévy process which admits a decomposition

$$Z_t = Z_{1,t} + Z_{2,t} + Z_{3,t} \tag{1.2}$$

in three independent Lévy processes where $Z_{2,t}$ is a Brownian motion (or degenerate) and $Z_{1,t}$, $Z_{3,t}$ are jump processes. The Lévy measure of $Z_{1,t}$, which counts the jumps of the process, is concentrated on $(-\infty,0)$ and it is supported by $(0,\infty)$ for $Z_{3,t}$. In connection with Lévy processes we refer to Bertoin (1996).

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It is the purpose of this paper to establish invariance principles, strong almost sure convergence results and approximations in probability for sums of centered extremal processes

$$\sum_{i=1}^{k} X_{i:[nt]}^{(n)} - b_t, \quad 0 \leqslant t \leqslant 1, \tag{1.3}$$

in $D[0,1]$ where $X_{i:[nt]}^{(n)}$ is the ith smallest (sequential) order statistic of $X_{n,1},\dots,X_{n,[nt]}$ given by the nth row and b_t are centering functions. The strong convergence results are obtained by the quantile approach for a special construction of order statistics. For instance, it is shown that under mild regularity assumptions a finite sum of extremal processes of form (1.3) approximates the whole process $Z_{1,t}$ in $D[0,1]$ in probability. Similary, the process $Z_{3,t}$ can be approximated by an appropriate finite sum of upper extremal processes which asymptotically become independent of (1.3). The method of proof is based on the quantile approach and an almost sure limit theorem for the joint distribution of extremal processes, see Section 3. The distributional convergence of extremal processes for sequential maxima was earlier obtained by Dwass (1964,1966), Lamperti (1964), Resnick and Rubinovitch (1973) and Resnick (1987).

The present results are closely connected to known infinite series representations of Lévy processes without Gaussian part. We will give some references. The roots of the series representations of Section 4 go back to P. Lévy. His results were summarized by Kahne (1995) who translated the work in a modern language, see also Bretagnolle (1973) for Lévy's decomposition.

Series for real independent increment processes have been considered by Ferguson and Klass (1972) and Kallenberg (1974) proved almost sure convergence of the series in the uniform norm on $D[0,1]$. LePage (1981) and Rosinski (1990) considered series in Banach spaces. Rosinski (1990) mentioned the connection to the Ito–Lévy representation of independent increment processes and he quoted a paper of Resnick.

In connection with the series representation of infinitely divisible distributions (and related convergence results) there exists a huge amount of literature. For stable laws we refer to LePage et al. (1981), and Csörgő et al. (1986) who used a strong approximation, see also Csörgő and Horváth (1988). In this spirit infinitely divisible laws were treated by Csörgő et al. (1988). A good reference for further results is also the volume about sums, trimmed sums and extremes of Hahn et al. (1991). The role of sums of extreme order statistics in connection with convergent partial sums was discussed by Janssen (1989,1994) where the first paper established almost sure convergence results (and convergence in probability) to stable laws whereas the second paper introduced distributional convergence for sums of extremes of arbitrary non i.i.d. triangular arrays. Strong invariance principles for partial sums with stable limit laws in $D[0,1]$ were established by LePage et al. (1997). Ould-Rois (1991) considered an invariance principles for trimmed sums.

The accuracy of the series approximation of infinitely divisible random variables was studied by several authors. We refer to Csörgő (1989a,b, 1995), Janssen and Mason (1990) and Bentkus et al. (1996) where rates of convergence can be found.

The meaning of the present results will be explained in the next example.

Example 1.1 (Stable process with the index $\frac{1}{2}$). Let $((B(t))_{t\in[0,\infty)}$ be a standard Brownian motion on $C[0,\infty)$. Then

$$Z_t := \inf\{s\colon B(s) \geqslant t\}, \quad 0 \leqslant t \tag{1.4}$$

defines a Lévy process on $[0,\infty)$ which is a stable process with the index $\alpha = \frac{1}{2}$ of stability, see Feller (1971) for instance. Notice that Z_t has only positive jumps and we have $Z_t = Z_{3,t}$ according to (1.2).

Consider now a rowwise i.i.d. triangular array of non-negative random variables $X_{n,i}$ with

$$\sum_{i=1}^{n} X_{n,i} \to Z_1 \tag{1.5}$$

in distribution as $n \to \infty$. For instance, we can choose

$$X_{n,i} = Z_{i/n} - Z_{(i-1)/n} \quad \text{for } 1 \leqslant i \leqslant n \tag{1.6}$$

which are just the inter arrival times of the underlying Brownian motion which are determined by the levels $t_{i-1} = (i-1)/n$ and $t_i = i/n$. In this case it is pointed out that a finite number of the largest inter arrival times can be used to approximate the whole process $(Z_t)_{t\in[0,1]}$.

More precisely, we will construct in the situation of (1.5) versions of $X_{n,i}$ and Z_t such that the following result holds in the Skorohod space $(D[0,1],d)$ which is endowed with the Skorohod metric d. For each $\varepsilon_1, \varepsilon_2 > 0$ there exist non-negative integers k and n_0 such that

$$P\left(d\left(\left(\sum_{i=1}^{k} X^{(n)}_{[nt]+1-i:[nt]} \right)_t, (Z_t)_t \right) \geqslant \varepsilon_1 \right) \leqslant \varepsilon_2 \tag{1.7}$$

holds for all $n \geqslant n_0$, see (1.3) for the definition of sequential order statistics. Thus, a finite sum of extremal processes approximates the limit of the total partial sum in $D[0,1]$. Notice that the centering function b_t of (1.3) can here be ignored. Within this construction the following strong invariance principle will be proved for a representation of the scheme (1.6). For each sequence of increasing functions of integers $t \mapsto k_n(t)$ with $k_n(t) \to \infty$ for each $t \in (0,1]$, we have almost sure convergence of the largest $k_n(\cdot)$ extremal processes in $D[0,1]$, namely

$$d\left(\left(\sum_{i=1}^{k_n(t)} X^{(n)}_{[nt]+1-i:[nt]} \right)_t, \ (Z_t)_t \right) \to 0 \quad P \text{ a.e.} \tag{1.8}$$

The result follows from Lemma 4.4. Moreover, the order statistic $X^{(n)}_{[nt]+1-k:[nt]}$ of the nth row is almost surely convergent to the kth largest jump of the process $(Z_s)_{s\in[0,t]}$. All these properties reveal the meaning of the largest inter arrival times given by the special array (1.6).

The present paper is organized as follows. Section 2 establishes the quantile approach to extremal processes which is based on Rényi's representation of uniform order statistics. The joint strong convergence of sequential order statistics to extremal processes is studied in Section 3. It is shown that under certain conditions the lower and upper sequential order statistics become independent. In Section 4 new strong invariance principles for extreme sequential order statistics attracted to infinitely divisible laws are introduced. These results cover Example 1.1 above.

Throughout, let d be the Skorohod metric on $D[0,1]$ and $\|\cdot\|$ the norm of uniform convergence. Define $x \wedge a = \min(x,a)$ and $x \vee a = \max(x,a)$.

2. The quantile approach for the joint distribution of extremal processes

In this section almost sure convergence results for the joint distribution of sequential order statistics are introduced. The results rely on a well-known representation of the vector of order statistics which is now applied to functional limit theorems. Let us start with a sequence $U_1, U_2, \dots$ of i.i.d. uniformly distributed random variables on $(0,1)$. It is well-known that their order statistics

$$(U_{1:n}, \dots, U_{n:n}) \stackrel{\mathscr{D}}{=} \left(\frac{S_1}{S_{n+1}}, \dots, \frac{S_n}{S_{n+1}} \right) \tag{2.1}$$

are equal in distribution to quotients of partial sums where $S_k = \sum_{i=1}^k Y_i$ is the partial sum of an i.i.d. sequence of standard exponential random variables Y_i with mean 1, see Breiman (1968) or Reiss (1989, p. 41) for a recent reference. Our construction below can be explained as follows. Assume for a moment that the order statistics $U_{j:n}$ are known. Then the chronological order of the $U_{j:n}$ is given by the antiranks $(D_{n1}, \dots, D_{nn})$ with

$$U_{i:n} = U_{D_{ni}} \quad \text{for } i = 1, \dots, n.$$

The vector of antiranks is uniformly distributed on the set of permutations of $1, \dots, n$ and they are independent of the order statistics, see Hájek and Šidák (1967). On the other hand, if $(\sigma(i))_{i \leqslant n}$ denotes another uniformly distributed permutation mutually independent of the U's then

$$(U_1, \dots, U_n) \stackrel{\mathscr{D}}{=} (U_{\sigma(1):n}, \dots, U_{\sigma(n):n}) \tag{2.2}$$

are equal in distribution. This simple observation will be combined with (2.1) which gives us an appropriate random assignment for the treatment of sequential order statistics. In addition, let $V_1, V_2, \dots$ be a second sequence of i.i.d. uniformly distributed random variables on (0,1) mutually independent of the U's. For fixed $0 < t \leqslant 1$ consider the Bernoulli process

$$n \mapsto \sum_{j=1}^{n} 1_{[0,t)}(V_j)$$

which jumps at the integers $\tau_t^{(1)}, \tau_t^{(2)}, \dots$ given by

$$\tau_t^{(1)} = \min\{ j\colon 1_{[0,t)}(V_j) = 1\}$$

and

$$\tau_t^{(k+1)} = \min\{ j > \tau_t^{(k)}\colon 1_{[0,t)}(V_j) = 1\}. \tag{2.3}$$

Notice that $t \mapsto \tau_t^{(k)}$ is right continuous on $(0,1]$ for each k.

In the next step, let $\sigma_n(j)$ be the jth antirank of $V_{j:n}$, i.e. $V_{j:n} = V_{\sigma_n(j)}$. Thus,

$$(\bar{U}_1, \dots, \bar{U}_n) := \left(\frac{S_{\sigma_n(i)}}{S_{n+1}} \right)_{i \leqslant n} \stackrel{\mathscr{D}}{=} (U_1, \dots, U_n) \tag{2.4}$$

is another representation of the U's. For convenience define $V_{j:n} = 0$ for $j \leqslant 0$ and $V_{j:n} = 1$ if $j > n$ for uniform order statistics.

Lemma 2.1. *The joint distribution of the sequential order statistics of* $\bar{U}_1, \dots, \bar{U}_n$ *is given by*

$$\left(\bar{U}_{1:k}, \dots, \bar{U}_{k:k}\right)_{k \leqslant n} = \frac{1}{S_{n+1}} \left(S_{\tau^{(1)}_{V_{k+1:n}}}, \dots, S_{\tau^{(k)}_{V_{k+1:n}}} \right)_{k \leqslant n}. \tag{2.5}$$

Proof. Consider outcomes $v_1, \dots, v_n$ of the V_i. Choose t to be the order value $t = v_{k+1:n}$. Suppose that $\sigma_n(1) = i_1, \dots, \sigma_n(k) = i_k$ are the antiranks of the k lowest order statistics. Observe that for $j \leqslant n$ the relation $v_j < v_{k+1:n}$ holds iff $j \in \{i_1, \dots, i_k\}$. Thus,

$$\left(\tau^{(j)}_{v_{k+1:n}}\right)_{j \leqslant k} = \left(i_{j:k}\right)_{j \leqslant k}$$

are now determined to be the order values of $i_1, \dots, i_k$. Conditional under the v_i's we have

$$(\bar{U}_1, \dots, \bar{U}_k) = \frac{1}{S_{n+1}} (S_{i_1}, \dots, S_{i_k}).$$

Since the partial sums S_i are ordered by their index the lemma follows from the equation

$$(\bar{U}_{1:k}, \dots, \bar{U}_{k:k}) = \frac{1}{S_{n+1}} (S_{i_{1:k}}, \dots, S_{i_{k:k}}). \qquad \square$$

The construction above is the key of almost sure functional limit theorems for lower order statistics. However, we like to treat lower and upper order statistics simultaneously. Based on Lemma 2.1 we choose a further construction which yields the asymptotic independence of the lower and upper part of order statistics. Throughout, consider four mutually independent copies of sequences $Y'_i, \tilde{Y}_i, V'_i$ and $\tilde{V}_i$, $i \in \mathbb{N}$, of Y_i and V_i with related partial sums S'_k, $\tilde{S}_k$ and first entrance times $\tau'^{(k)}_t$, $\tilde{\tau}^{(k)}_t$ (2.3), respectively. Define a triangular system of partial sums

$$S_{k,n} = \sum_{i=1}^{k} Y'_i \quad \text{for } k \leqslant [n/2]$$

and

$$S_{k,n} = S_{[n/2],n} + \sum_{j=1}^{k-[n/2]} \tilde{Y}_{n+1-j-[n/2]} \quad \text{for } n \geqslant k > [n/2]. \tag{2.6}$$

This special construction of uniform order statistics based on (2.6) is due to Csörgő et al. (1986), see also Janssen (1989). Notice that

$$(U_{1:n}, \dots, U_{n:n}) \stackrel{\mathscr{D}}{=} \left(\frac{S_{1,n+1}}{S_{n+1,n+1}}, \dots, \frac{S_{n,n+1}}{S_{n+1,n+1}} \right) \tag{2.7}$$

holds where now the k largest order statistics

$$U_{n+1-k:n} \stackrel{\mathscr{D}}{=} 1 - \frac{\tilde{S}_k}{S_{n+1,n+1}} \tag{2.8}$$

is mainly determined by $\tilde{S}_k$ for $k < [(n+1)/2]$.

At this stage the random variables V_i, $1 \leqslant i \leqslant n$, are substituted by a new triangular array $V_{i,n}$ defined by

$$V_{i,n} = V_i' \quad \text{for } i \leqslant [n/2]$$

and

$$V_{i,n} = \tilde{V}_{n+1-i} \quad \text{for } [n/2] < i \leqslant n. \tag{2.9}$$

As above let $\sigma_n(\cdot)$ be the antiranks of $V_{1,n}, \ldots, V_{n,n}$. Then

$$\begin{aligned}(\bar{U}_1, \ldots, \bar{U}_n) &:= \frac{1}{S_{n+1,n+1}} \left(S_{\sigma_n(i),n+1}\right)_{i \leqslant n} \\ &\stackrel{\mathscr{D}}{=} (U_1, \ldots, U_n)\end{aligned} \tag{2.10}$$

holds.

If we are concerned with sequential order statistics of a triangular array we will introduce an extra index n on the top for the order statistics $\bar{U}_{k:m}^{(n)}$ which is by definition the kth-order statistic among $\bar{U}_1, \ldots, \bar{U}_m$ of the nth row.

The path of our process will lie in the cad lag space $D(0,1]$ which is defined on the left-sided open interval $(0,1]$. The common Skorohod topology for compact intervals is extended on $D(0,1]$ in the sense of Resnick (1987, Section 4.4). It is quite obvious that a sequence of monotone functions $x_n(\cdot)$ is convergent to $x_0(\cdot)$ in $D(0,1]$ if $x_n(t) \to x_0(t)$ holds for all continuity points t of $x_0(\cdot)$. The space $D(0,1]^{\mathbb{N}}$ is equipped with the product topology.

Theorem 2.1. *The joint sequential order statistics of* (2.10)

$$t \mapsto \left((n\bar{U}_{k:[nt]}^{(n)})_{k \in \mathbb{N}}, (n(1 - \bar{U}_{[nt]+1-j:[nt]}^{(n)}))_{j \in \mathbb{N}}\right) \tag{2.11}$$

are almost convergent in $D(0,1]^{\mathbb{N}} \times D(0,1]^{\mathbb{N}}$ *to the random variable*

$$t \mapsto \left((S'_{\tau_t'^{(k)}})_{k \in \mathbb{N}}, (\tilde{S}_{\tilde{\tau}_t^{(j)}})_{j \in \mathbb{N}}\right). \tag{2.12}$$

Proof. For the sequel let $V_{i:n}$ denote the order statistics of the scheme $V_{1,n}, \ldots, V_{n,n}$, defined in (2.9), and let $\tau_{V_{j:n}}^{(k)}$ be their kth entrance time at random time $t = V_{j:n}$. First the kth lower coordinate will be considered in $D[a,1]$ for $a > 0$. The proof uses the random time transformation

$$t \mapsto V_{[nt]+1:n}$$

which satisfies

$$\sup_{t \in [0,1]} |V_{[nt]+1:n} - t| \to 0 \quad P \text{ a.e} \tag{2.13}$$

by the Glivenko Cantelli theorem. As a consequence, we see that it is enough to prove almost sure convergence of our monotone functions at the random times $V_{[nt]+1:n}$

instead of t. For this purpose, we will first compare the kth lower order statistic with the members of the stopped partial sum $S_{j:n+1}$ (2.6). In this situation we have by Lemma 2.1 for $n \geqslant 1/a$ and $k \leqslant n$

$$\sup_{t\in[a,1]}\left|n\bar{U}^{(n)}_{k:[nt]} - S_{\tau^{(k)}_{V_{[nt]+1:n}},n+1}\right| = \sup_{t\in[a,1]}\left|S_{\tau^{(k)}_{V_{[nt]+1:n}},n+1}\left(\frac{n}{S_{n+1,n+1}}-1\right)\right|$$
$$\leqslant \left|S_{\tau^{(k)}_{V_{[na]+1:n}},n+1}\right|\left|\frac{n}{S_{n+1,n+1}}-1\right|. \tag{2.14}$$

Obviously, the first entrance times $\tau'^{(k)}_t$ of our sequence V'_j are finite with probability 1 for each $t>0$. In comparison with the mixed situation (2.9) our construction yields $\tau^{(k)}_t = \tau'^{(k)}_t$ finally again with probability 1. Notice that now the event

$$\left\{S_{\tau^{(k)}_{V_{[na]+1:n}},n+1} \to \infty\right\} \tag{2.15}$$

has probability zero and

$$\sup_{t\in[a,1]}\left|S_{\tau^{(k)}_{V_{[nt]+1:n}},n+1} - S'_{\tau'^{(k)}_{V_{[nt]+1:n}}}\right| \to 0 \tag{2.16}$$

holds almost surely. The strong law of large numbers together with (2.15) proves that (2.14) converges almost surely to zero. The almost sure convergence of the lower order statistics now follows from (2.13) and (2.16).

The key of the treatment of the jth largest order statistic is relation (2.8) which allows a reduction of the proof to the first case. Introduce the permutation $\rho_n(i) = (n+1)-\sigma_n(i)$ where for instance $\rho_n(i)$ denotes the antirank of $V_{n+1-i,n} = \tilde{V}_i$ among $V_{1,n},\dots,V_{n,n}$ provided $i > [(n+1)/2]$ holds. Our construction (2.10) yields

$$(1-\bar{U}_1,\dots,1-\bar{U}_n) = \frac{1}{S_{n+1,n+1}}\left(S_{n+1,n+1} - S_{n+1-\rho_n(i),n+1}\right)_{i\leqslant n}, \tag{2.17}$$

where again $S_{n+1,n+1} - S_{n+1-\rho_n(i),n+1} = \sum_{j=1}^{\rho_n(i)} \tilde{Y}_j$ holds for $\rho_n(i) \leqslant [n/2]$.

A moments reflection yields that (2.10) and (2.17) have the same structure if the index prime $'$ and tilde $\sim$ are changed. Thus the proof of the lower part also implies the result for the upper order statistics. □

3. Almost sure convergence of extremal processes and point processes of extremes

The preceding results apply to triangular arrays of random variables. Sequential results for order statistics are included if we set $a \in \{\pm\infty\}$ below.

Suppose that

$$X_{n,1},\dots,X_{n,n} \tag{3.1}$$

is a triangular array of row-wise i.i.d. random variables with joint distribution function F_n on $\mathbb{R}$ and sequential order statistics $X^{(n)}_{1:k} \leqslant \cdots \leqslant X^{(n)}_{k:k}$ of $X_{n,1},\dots,X_{n,k}$.

We will use some assumptions.

Let $a \in [-\infty,\infty]$ be fixed. Assume that there exists a right continuous increasing function $G_1 : (-\infty,a) \to \mathbb{R}$ with $G_1(x) \downarrow 0$ as $x \downarrow -\infty$ and

$$nF_n(x) \to G_1(x) \tag{3.2}$$

for all continuity points $x \in (-\infty, a)$ of G_1. Notice that condition (3.2) is equivalent to the convergence of

$$X_{1:n}^{(n)} \wedge a \to Z_1 \tag{3.3}$$

in distribution where Z_1 is a random variable with distribution function

$$P(Z_1 \leqslant x) = (1 - \exp(-G_1(x)))1_{(-\infty,a)}(x) + 1_{[a,\infty)}(x). \tag{3.4}$$

Assume also that there exists a second right continuous decreasing function $G_2 : (a, \infty) \to \mathbb{R}$ with $G_2(x) \to 0$ as $x \to \infty$ and

$$n(1 - F_n(x)) \to G_2(x) \tag{3.5}$$

again for all continuity points $x \in (a, \infty)$ of G_2. As above (3.5) is equivalent to the distributional convergence of $X_{n:n}^{(n)} \vee a \to Z_2$ where

$$P(Z_2 < x) = \exp(-G_2(x))1_{(a,\infty)}(x) \tag{3.6}$$

holds.

The present approach now establishes almost sure convergence results for extremes given by the special choice of random variables

$$X_{i,n}^{(n)} := F_n^{-1}(\bar{U}_i), \tag{3.7}$$

where $(\bar{U}_1, \dots, \bar{U}_n)$ is as in scheme (2.10). For this purpose, we introduce similarly to Janssen (1994, p. 1768) the monotone inverse functions $\psi_1 : (0, \infty) \to (-\infty, a]$ and $\psi_2 : (0, \infty) \to [a, \infty)$ of G_1 and G_2 by

$$\psi_1(y) = \inf\{t\colon G_1(t) \geqslant y\} \wedge a, \tag{3.8}$$

$$\psi_2(y) = \inf\{t\colon G_2(t) \leqslant y\} \vee a, \tag{3.9}$$

where ψ_1 and ψ_2 are left continuous.

In contrast to Theorem 2.1 the present sequential results can also be used to establish convergence on $D[0,1]$ in various cases. We will see that under our assumptions

$$X_{n,1} \to a \tag{3.10}$$

holds almost surely. Thus we define $X_{k:0} = a$, $\psi_1(\infty) = \psi_2(\infty) = a$ and set $\tau_0^{\prime(k)} = \tilde{\tau}_0^{(j)} = \infty$ and $S'_\infty = \tilde{S}_\infty = \infty$.

Theorem 3.1. *For the sequential order statistics of* (3.7) *we have almost sure convergence of*

$$t \mapsto \left((X_{k:[nt]}^{(n)})_{k\in\mathbb{N}}, (X_{[nt]+1-j:[nt]}^{(n)})_{j\in\mathbb{N}}\right) \tag{3.11}$$

in $D(0,1]^{\mathbb{N}} \times D(0,1]^{\mathbb{N}}$ *to the random variable*

$$t \mapsto \left((\psi_1(S'_{\tau_t^{\prime(k)}}))_{k\in\mathbb{N}}, (\psi_2(\tilde{S}_{\tilde{\tau}_t^{(j)}}))_{j\in\mathbb{N}}\right) \tag{3.12}$$

where the lower ($k \in \mathbb{N}$) *and upper parts* ($j \in \mathbb{N}$) *of* (3.12) *are independent. If in addition* $a \in \mathbb{R}$ *holds we have almost sure convergence of* (3.11) *in* $D[0,1]^{\mathbb{N}} \times D[0,1]^{\mathbb{N}}$.

Proof. Throughout, the almost sure convergence result will be proved for fixed k and j. Suppose that $a > -\infty$ holds. Then we have

$$X_{k:[nt]}^{(n)} = F_n^{-1}(n\bar{U}_{k:[nt]}/n). \tag{3.13}$$

Now the present proof follows along the lines of Janssen (1994, p. 1769). The inverse

$$\psi_{1,n}(x) := F_n^{-1}(x/n) \wedge a \tag{3.14}$$

of $nF_{n|(-\infty,a]}$ converges by (3.2) pointwise to ψ_1 except for a countable subset $N \subset (0,\infty)$. Since each partial sum S'_m has continuous distribution the set

$$M := \left\{ S'_{\tau_t'^{(k)}} \in N \text{ for some } t \in (0,1] \right\}$$

has probability zero. If we combine this result with the almost sure convergence result for $n\bar{U}_{k:[nt]}$ of Theorem 2.1 we have almost sure convergence of

$$\psi_{1,n}(n\bar{U}_{k:[nt]}^{(n)}) \to \psi_1(S'_{\tau_t'^{(k)}}) \tag{3.15}$$

on $(0,1]$ with probability 1, where again (3.15) converges on M^c except for a countable number of t's. In this context the properties of the convergence of monotone functions for all continuity points are repeatedly used. Thus (3.14) and (3.15) yield the convergence of the decreasing functions

$$\left(X_{k:[nt]}^{(n)} \wedge a \right)_{t\in(0,1]} \to \left(\psi_1(S'_{\tau_t'^{(k)}}) \right)_{t\in(0,1]} \tag{3.16}$$

for all continuity points (of the t's) of the right-hand side with probability 1.

In the next step, we will show how (3.10) and convergence in $D[0,1]$ can be proved.

Notice that $V_{m(n):n} \to 0$ holds a.e. for some sequence $m(n) \to \infty$, $m(n)/n \to 0$, which implies $\sigma_n(1) \to \infty$ a.e.. Thus $\tau_t'^{(k)} \to \infty$ follows a.e. as $t \downarrow 0$. Since the functions (3.16) are decreasing in t we have

$$\liminf_{n\to\infty} X_{n,1} \geqslant \lim_{t\downarrow 0} \psi_1(S'_{\tau_t'^{(k)}}) = a \quad \text{a.e.} \tag{3.17}$$

since $X_{n,1} \geqslant X_{k:[nt]}^{(n)}$ holds for $t \geqslant k/n$. The same arguments apply to the array $-X_{n,i}$ if $a \in \mathbb{R}$. Thus (3.17) implies (3.10) and thus the extra condition $(\wedge a)$ on the left-hand side of (3.16) can be cancelled. If $a \in \mathbb{R}$ holds this result implies the desired convergence in $D[0,1]$ since (3.16) holds for all continuity points t on a set with probability 1.

If $a < \infty$ holds the upper order statistics can be treated similarly for fixed j by a minor modification of the proof above concerning inverse functions. We will use the following elementary fact of inverse functions. Let X be a random variable with ordinary left continuous inverse distribution function F_X^{-1} in the sense of (3.8). Then

$$\tilde{F}_X^{-1}(y) = \sup\{t:\ P(X < t) \leqslant y\}, \tag{3.18}$$

$0 < y < 1$, defines another right continuous inverse.

It is easy to check that

$$-F_X^{-1}(1-y) = \tilde{F}_{-X}^{-1}(y) \tag{3.19}$$

holds and again convergence in distribution implies convergence of the $\tilde{F}$-inverse functions for all continuity points. Keeping (3.19) in mind we will consider the scheme

$-X_{i,n}$ with joint distribution functions $H_n(x) := 1 - F_n((-x)-)$. The "$\sim$" inverse of nH_n is just

$$y \mapsto \tilde{H}^{-1}(y/n) = -F_n^{-1}(1 - y/n) \tag{3.20}$$

if (3.19) is applied. Since $nH_n(x) \to G_2(-x)$ holds for all continuity points of G_2 with $-x > a$ we have convergence of the inverse functions

$$\tilde{H}^{-1}(\cdot/n) \wedge (-a) \to -\psi_2(\cdot) \tag{3.21}$$

on the set of continuity points of ψ_2. Alltogether we see that

$$\begin{aligned} -X^{(n)}_{[nt]+1-j:[nt]} &= -F_n^{-1}\left(U^{(n)}_{[nt]+1-j:[nt]}\right) \\ &= -F_n^{-1}\left(1 - n(1 - U_{[nt]+1-j:[nt]})/n\right) \to -\psi_2(\tilde{S}_{\tilde{\tau}_t^{(j)}}) \end{aligned} \tag{3.22}$$

is almost everywhere convergent in $D(0,1]$ if Theorem 2.1, (3.10), (3.20) and (3.21) are taken into account. As above the monotony of (3.22) together with (3.10) implies the convergence on $D[0,1]$. The convergence for each k and each j implies the joint convergence of the vectors with respect to the product topology. □

Remark 3.1. The quantile function ψ_2 of (3.9) can obviously be substituted by its right continuous version

$$\tilde{\psi}_2(y) := \sup\{t: G_2(t-) \geqslant y\}. \tag{3.23}$$

Notice that then

$$\psi_2\left(\tilde{S}_{\tilde{\tau}_t^{(j)}}\right) = \tilde{\psi}_2\left(\tilde{S}_{\tilde{\tau}_t^{(j)}}\right) \tag{3.24}$$

holds a.e. for each t and all $j \in \mathbb{N}$.

Example 3.1 (Convergence of extremes)**.** Suppose that (3.2) or equivalently (3.3) holds for $a=\infty$. Then Theorem 3.1 proves almost sure convergence of the kth extremal process (with respect to the min-operation)

$$\left(X^{(n)}_{k:[nt]}\right)_{t\in(0,1]} \to \psi_1\left(S'_{\tau_t'^{(k)}}\right)_{t\in(0,1]} \tag{3.25}$$

in $D(0,1]$. For $k=1$ this is an almost sure version of the functional limit theorem of Resnick (1987, Proposition 4.20), who considered maxima instead of lower order statistics. Distributional convergence of (3.25) can easily be extended to $D(0,b]$ for each $b>0$ and $D(0,\infty)$ if we start with an unbounded sequence of i.i.d. variables in each row. Define now $\xi_t = S'_{\tau_t'^{(1)}}$ for each $t \in (0,1]$. Then $(\xi_t)_{t\in(0,1]}$ yields an explicit construction of an extremal process relative on the time domain $(0,1]$. Notice that ξ_t has the survival function

$$P(\xi_t > x) = \exp(-tG_1(x)) \tag{3.26}$$

and ξ_t is a process with "independent and stationary increments" with respect to the min-operation. To explain this let $0 = t_0 < t_1 < t_2 < \cdots < t_n \leqslant 1$ be a finite set of coordinates and introduce further mutually independent random variables $W_1, \ldots, W_n$ with distribution

$$W_i \overset{\mathscr{D}}{=} \xi_{t_i - t_{i-1}} \quad \text{for } i = 1, \ldots, n.$$

Then

$$(\xi_{t_1},\ldots,\xi_{t_n}) \stackrel{\mathscr{D}}{=} \left(\min_{i\leqslant j} W_i\right)_{j=1,\ldots,n} \tag{3.27}$$

holds. We refer to Resnick and Rubinovitch (1973) who studied the structure of extremal processes.

Remark 3.2. Extremal processes can be used to motivate and introduce the popular class of proportional hazard models for continuous time in survival models of statistics, see (3.26). They can be motivated by the construction principle (3.27) as follows. If the time t_j is increasing the survival time

$$\xi_{t_j} = \min(W_1,\ldots,W_j)$$

is limited by an increasing number of risks which are expressed by independent random variables $W_1,\ldots,W_j$. Within this model, the survival time ξ_t is determined by a sequential accumulation of risk factors.

It is well known that there is a strong connection between convergence of extremes and weak convergence of point processes. Since the sequential versions of these results are needed in Section 4 the connection is briefly summarized and outlined. Throughout, the background and terminology of Resnick (1987) is used. Motivated by Theorem 2.1 let us consider the empirical point process

$$N_n^{(t)} = \sum_{k=1}^{[nt]} \varepsilon_{n\bar{U}_k} = \sum_{k=1}^{[nt]} \varepsilon_{nU^{(n)}_{k:[nt]}} \tag{3.28}$$

for $0<t\leqslant 1$ on the state space $(0,\infty)$. Its intensity measure $t\lambda_{|(0,n)} \to t\lambda_{|(0,\infty)}$ is convergent where λ denotes the Lebesgue measure. It is well known that $N_n^{(t)}$ is weakly convergent to a Poisson point process $N^{(t)}$ with intensity measure $t\lambda_{|(0,\infty)}$, see Resnick (1987). Our Theorem 2.1 now implies weak convergence of $N_n^{(t)}$ to the special construction

$$N^{(t)} = \sum_{k=1}^{\infty} \varepsilon_{S'_{\tau_t'(k)}} = \sum_{j=1}^{\infty} \varepsilon_{S'_j} 1_{(0,t)}(V'_j), \tag{3.29}$$

where $N^{(t)}$ can be understood as a thinned Poisson process obtained from the standard Poisson process $N^{(1)}$ given by its renewal form.

The family of Poisson point processes (3.29) can be embedded into a new Poisson point process M with state space $E=(0,1)\times(0,\infty)$. According to Proposition 3.8 of Resnick (1987)

$$M = \sum_{k=1}^{\infty} \varepsilon_{(V'_k,S'_k)} \tag{3.30}$$

defines another Poisson point process with intensity measure $\nu = \lambda_{|(0,1)} \otimes \lambda_{|(0,\infty)}$ on E. If the transformation ψ_1 (3.8) is applied to the second coordinate of M the transformed point process

$$M^{\psi_1} = \sum_{k=1}^{\infty} \varepsilon_{(V'_k,\psi_1(S'_k))} 1_{(0,1)\times(-\infty,a)}(V'_k,\psi_1(S'_k)) \tag{3.31}$$

is again Poissonian with intensity measure $\nu' = \lambda_{|(0,1)} \otimes \mu$ on the state space $(0,1) \times (-\infty, a)$ where μ is the measure defined by the measure generation function $G_{1|(-\infty,a)}$, see (3.2). As stated in (3.29) the Poisson point process

$$M^{\psi_1}((0,t) \times \cdot) = \sum_{k=1}^{\infty} \varepsilon_{\psi_1(S'_k)}((-\infty, a) \cap \cdot) 1_{(0,t)}(V'_k) \tag{3.32}$$

is again the weak limit of the sequential partial sums which are formed by the array (3.1).

Obviously, a second Poisson point process

$$M^{\psi_2} = \sum_{j=1}^{\infty} \varepsilon_{(\tilde{V}_j, \psi_2(\tilde{S}_j))} 1_{(0,1)\times(a,\infty)}(\tilde{V}_j, \psi_2(\tilde{S}_j)) \tag{3.33}$$

can be established on the state space $(0,1) \times (a, \infty)$ for the upper extremes, which is independent of (3.31).

As a further application of Theorem 2.1 we mention that Theorem 2.1 also applies to the weak convergence of the marked point process

$$\sum_{k=1}^{[nt]} \varepsilon_{\left(k, nU^{(n)}_{k:[nt]}\right)} \quad \text{to} \quad \sum_{k=1}^{\infty} \varepsilon_{\left(k, S'_{\tau'(k)_k}\right)} \tag{3.34}$$

on the state space $\mathbb{N} \times (0, \infty)$.

4. Convergent sums of sequential order statistics in $D[0,1]$ and Lévy processes

In this section let us consider a rowwise i.i.d. triangular array of random variables (3.1) with convergent partial sums (1.1). Again F_n denotes the distribution function of $X_{n,1}$. A necessary condition for the convergence of the partial sums is the convergence of

$$nF_n(x) \to \eta(-\infty, x] \quad \text{for } x < 0$$

and

$$n(1 - F_n(x-)) \to \eta([x, \infty)) \quad \text{for } x > 0 \tag{4.1}$$

for all continuity points x of the Lévy measure η on $\mathbb{R}\backslash\{0\}$, see Gnedenko and Kolmogorov (1968, p. 116). It satisfies

$$\int \min(1, \|x\|^2)\, \mathrm{d}\eta(x) < \infty \tag{4.2}$$

and it is well known that the Lévy measure η counts the number of jumps of the Lévy process in the following sense: For each Borel set $A \subset \mathbb{R}$ with $\eta(\partial A) = 0$ and $0 \notin \bar{A}$ we have

$$\eta(A) = E\left(\sum_{0 \leqslant t \leqslant 1} 1_A(Z_t - Z_{t-}) \right), \tag{4.3}$$

see for instance Gihman and Skorohod (1979). After a linear centering procedure in time t (which is always assumed below) the characteristic functions of the Lévy process Z_t are given by the Lévy–Hinčin formula $\varphi_{Z_t}(y) = \exp(t\rho(y))$, $y \in \mathbb{R}$, where

$$\rho(y) = \rho(y, \sigma^2, \eta)$$
$$= -\frac{\sigma^2 y^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (\exp(\mathrm{i}xy) - 1 - \mathrm{i}xy\ 1_{(-\beta,\beta)}(x))\,\mathrm{d}\eta(x) \tag{4.4}$$

is the exponent and $\sigma^2 \geqslant 0$ is the variance of the normal part. We will assume that the truncation points $\pm\beta$ are continuity points of the (possibly unbounded) distribution function (4.1) of η with $\beta > 0$. The decomposition of the process (1.2) can now be described in terms of characteristic functions, namely by

$$\rho(y) = \rho_1(y) + \rho_2(y) + \rho_3(y), \tag{4.5}$$

where by definition $\rho_1(y) = \rho(y, 0, \eta_{|(-\infty,0)})$, $\rho_2(y) = \rho(y, \sigma^2, 0)$ and, $\rho_3(y) = \rho(y, 0, \eta_{|(0,\infty)})$. The process $Z_{j,t}$ corresponds to $\exp(t\rho_j)$ for $j \leqslant 3$.

As mentioned in the introduction the processes $Z_{1,t}$ and $Z_{3,t}$ admit a series representation given by our random variables and point processes introduced in Sections 2 and 3. If we define $G_1(x) = \eta(-\infty, x]$ for $x < 0$ and $G_2(x) = \eta(x, \infty)$ for $x > 0$ and set $a = 0$ then the inverse distribution functions (3.8) and (3.9) determine the Ferguson and Klass (1972) series representations

$$Z_{1,t} \stackrel{\mathscr{D}}{=} Z'_{1,t} := \sum_{j=1}^{\infty} (\psi_1(S'_j) 1_{(0,t)}(V'_j) - tE(\psi_1(S'_j) 1_{(-\beta,0)}(\psi_1(S'_j)))), \tag{4.6}$$

$0 \leqslant t \leqslant 1$, and

$$Z_{3,t} \stackrel{\mathscr{D}}{=} \tilde{Z}_{3,t} := \sum_{j=1}^{\infty} (\psi_2(\tilde{S}_j) 1_{(0,t)}(\tilde{V}_j) - tE(\psi_2(\tilde{S}_j) 1_{(0,\beta)}(\psi_2(\tilde{S}_j)))), \tag{4.7}$$

$0 \leqslant t \leqslant 1$, which is almost surely convergent in $D[0,1]$ with respect to the sup-norm, see Kallenberg (1974). A univariate discussion is included in Janssen (1994). Throughout, the processes $Z_{1,t}$ and $Z_{3,t}$ will always be identified with the right-hand side of (4.6) and (4.7). Also the triangular array is always choosen according to the quantile representation (3.7) and (2.10). Thus, we see that $\psi_2(\tilde{S}_j)$ is the jth largest jump of the process $t \mapsto Z_{3,t}$, $t \in [0,1]$, which occurs at time $\tilde{V}_j$. The meaning of Theorem 3.1 can now be summarized as follows.

Remark 4.1. The kth lower order statistics and the jth largest order statistic $(X_{k:n}^{(n)}, X_{n+1-j:n}^{(n)})$ of the triangular array are almost surely convergent to $(\psi_1(S'_k), \psi_2(\tilde{S}_j))$ which are just the kth largest negative jump and the jth largest positive jump of the process Z_t for $t \in [0,1]$. The same assertion holds for the sequential order statistics if the process is restricted to $0 \leqslant s \leqslant t$.

The present results can now be applied in order to get new invariance principles for sums of sequential order statistics. Observe that in case

$$\int_{(-1,\infty)} |x|\,\mathrm{d}\eta(x) < \infty \tag{4.8}$$

the centering constants of (4.6) are convergent and they may be cancelled. Thus

$$\tilde{Z}_{1,t} := \sum_{j=1}^{\infty} \psi_1(S_j')1_{(0,t)}(V_j') \tag{4.9}$$

is again almost surely convergent in $D[0,1]$ with respect to the sup-norm. On the other hand, each Lévy process on $(-\infty,0]$ is up to a linear drift of type (4.9) and (4.8) holds with $\sigma=0$ and $\eta_{|(0,\infty)}=0$. The characteristic function of (4.9) is given by

$$\varphi_{\tilde{Z}_{1,t}}(y) = \exp\left(t\int_{\mathbb{R}\setminus\{0\}} (\exp(\mathrm{i}xy)-1)\,\mathrm{d}\eta(x)\right). \tag{4.10}$$

Theorem 4.1. *Let $X_{n,i}\leqslant 0$ be an i.i.d. triangular array with distributional convergent partial sum and limit characteristic function $\varphi_{\tilde{Z}_{1,1}}$ (4.10). Within our special representation (3.7) the following result holds. For each $t\in(0,1]$ let $1\leqslant k_n(t)\leqslant [nt]$ be any sequence with $k_n(t)\to\infty$ as $n\to\infty$ such that $t\mapsto k_n(t)$ is increasing for fixed $n\in\mathbb{N}$. Then*

$$d\left(\sum_{j=1}^{k_n(\cdot)} X_{j:[n\cdot]}^{(n)}, \tilde{Z}_{1,\cdot}\right) \to 0 \tag{4.11}$$

is convergent to zero in probability on the Skorohod space $(D[0,1],d)$. If in addition the partial sum $\sum_{i=1}^{[nt]} X_{i:[nt]}^{(n)} \to \tilde{Z}_{1,t}$ is almost surely convergent for each t within the special representation (3.7) then (4.11) is also almost surely convergent in $D[0,1]$.

Proof. Introduce the random variable

$$Y_t = \limsup_{n\to\infty} \sum_{j=1}^{k_n(t)} X_{j:[nt]}^{(n)}.$$

This process has decreasing trajectories. For fixed $t\in(0,1]$ we have by (2.14) and the proof of Theorem 3.1 the inequality

$$Y_t \leqslant \tilde{Z}_{1,t} \tag{4.12}$$

almost surely. The combination of the Theorems 2.1 and 3.1 of Janssen (1994) proves that

$$\sum_{j=1}^{k_n(t)} X_{j:[nt]}^{(n)} \to \tilde{Z}_{1,t} \tag{4.13}$$

is convergent in distribution as $n\to\infty$ for fixed t. Notice that (4.12) implies

$$P\left(-\sum_{j=1}^{k_n(t)} X_{j:[nt]}^{(n)} \leqslant -\tilde{Z}_{1,t}-\varepsilon\right) \to 0 \tag{4.14}$$

for each $\varepsilon>0$ and thus Lemma 4.1 below together with Remark 4.2 yield convergence in probability of (4.13) again for fixed t.

The proof of (4.11) now follows from routine convergence arguments for processes with decreasing trajectories. Let $M\subset[0,1]$, $0,1\in M$ be a countable dense subset.

For each subsequence there exists a further subsequence $\{m\} \subset \mathbb{N}$, such that (4.13) is almost surely convergent along $\{m\}$ for all $t \in M$. Consequently, (4.11) tends to zero almost surely along the subsequence $\{m\}$ and the desired result is proved.

If in addition the partial sums are almost surely convergent we obtain in view of (4.12)

$$\sum_{i=1}^{k_n(t)} X_{i:[nt]}^{(n)} \to \tilde{Z}_{1,t} \tag{4.15}$$

almost surely for fixed t. As above (4.15) implies almost sure convergence first on M and then of (4.11) as $n \to \infty$. □

Lemma 4.1. *Let W_n, $n \geqslant 0$, be a sequence of real random variables with $P(W_n \leqslant W_0 - \varepsilon) \to 0$ for each $\varepsilon > 0$. Suppose that there exists a strictly increasing function $f : \mathbb{R} \to [0, \infty)$ with continuous inverse and $|f(x) - f(y)| \leqslant |x - y|$ for all $x, y \in \mathbb{R}$. Let $\int f(W_0)\,\mathrm{d}P$ be finite and suppose that*

$$\int f(W_n)\,\mathrm{d}P \to \int f(W_0)\,\mathrm{d}P \tag{4.16}$$

holds as $n \to \infty$. Then $W_n \to W_0$ is convergent in P-probability.

Proof. It is enough to prove that $f(W_n) \to f(W_0)$ is convergent in probability. The assumptions imply that

$$P(f(W_n) - f(W_0) \leqslant -\varepsilon) \to 0$$

holds as $n \to \infty$ for each $\varepsilon > 0$. Now introduce new random variables $Y_n = \min(f(W_n), f(W_0) + \varepsilon/2)$. There exists a sequence $\delta_n \downarrow 0$ with

$$P(A_n) \to 0,$$

where A_n is the event $A_n = \{f(W_n) - f(W_0) \leqslant -\delta_n\}$. Since $\int Y_n 1_{A_n}\,\mathrm{d}P \to 0$ holds our assumptions imply

$$\int Y_n\,\mathrm{d}P \to \int f(W_0)\,\mathrm{d}P$$

as $n \to \infty$ if the inequality

$$\int f(W_0)\,\mathrm{d}P \leqslant \liminf_{n\to\infty} \int Y_n\,\mathrm{d}P_0 \leqslant \lim_{n\to\infty} \int f(W_n)\,\mathrm{d}P_0$$

is taken into account. Thus

$$\begin{aligned} &\varepsilon P(f(W_n) \geqslant f(W_0) + \varepsilon)/2 \\ &\qquad \leqslant \int_{\{f(W_n) \geqslant f(W_0)+\varepsilon\}} (f(W_n) - f(W_0) - \varepsilon/2)\,\mathrm{d}P \\ &\qquad \leqslant \int (f(W_n) - Y_n)\,\mathrm{d}P \to 0 \end{aligned}$$

proves the result. □

Remark 4.2. Condition (4.16) obviously holds for some function f if $W_n \to W_0$ is convergent in distribution. Under the assumptions of Lemma 4.1 distributional convergence implies the convergence in probability.

Remark 4.3. The assumptions of Theorem 4.1 allow different choices of the portion $k_n(t)$ of contributing lower order statistics. Note that for $k_n(t) = [nt]$ we obtain an invariance principle (in probability) for the total partial sum. On the other hand, if $k_n \leqslant n$ denotes a further sequence with $k_n \to \infty$ we may choose either $k_n(t) = k_n \wedge [nt]$ or $k_n(t) = [k_n t]$.

Next let us consider arbitrary convergent partial sums. Then it is known that the distributional convergence of sums of centered extremes for fixed $t > 0$ strongly depends on the portion $k_n(t)$ of order statistics since a non-trivial Gaussian part of the limit law changes the situation of (4.11) completely, see Janssen (1994). As conclusion of that paper recall that always the partial sum of a finite number of centered lower order statistics approximates the distribution of $Z_{1,t}$ up to ε within the Lévy metric for convergence in distribution. An approximation of this type also holds for extremal processes of the triangular array and the parts $Z_{1,t}$ and $Z_{3,t}$ of the Lévy process in $D[0,1]$.

Consider below distributional convergent partial sums (1.1). Without restriction we may assume that $E(X_{n,1} 1_{(-\beta,\beta)}(X_{n,1})) = 0$ holds for each n. (Otherwise the triangular array may be centered.) Then the characteristic function of the Lévy process is just (4.4) and (4.6), (4.7) provide a series representation of (1.2) for $Z'_{1,t}$, $\tilde{Z}_{3,t}$, respectively. In addition, assume that there exists some $0 < \alpha < 1$ with

$$\int \min(1, |x|^{2-\alpha})\, d\eta(x) < \infty. \tag{4.17}$$

Note for instance that for each stable non-Gaussian process there exists some α such that the latter condition holds.

Theorem 4.2. *Under the present assumptions about the underlying triangular array our construction* (3.7) *yields the following results in* $(D[0,1], d)$.

(a) (*One-sided version*) *For each pair of positive values* $\varepsilon_1, \varepsilon_2$ *there exist positive integers* k *and* n_0 *such that*

$$P\left(d\left(\sum_{j=1}^{k} \left(X^{(n)}_{j:[n\cdot]} - E\left(X^{(n)}_{j:[n\cdot]} 1_{(-\beta,\beta)}\left(X^{(n)}_{j:[n\cdot]} \right) \right) \right), Z'_{1,\cdot} \right) \geqslant \varepsilon_1 \right) \leqslant \varepsilon_2 \tag{4.18}$$

holds for all $n \geqslant n_0$.

(b) (*Two-sided version*) *For each pair of positive values* $\varepsilon_1, \varepsilon_2$ *there exist positive integers* k, r *and* n_0 *such that*

$$P\Bigg(d\Bigg(\sum_{j=1}^{k} (X^{(n)}_{j:[n\cdot]} - E(X^{(n)}_{j:[n\cdot]} 1_{(-\beta,\beta)}(X^{(n)}_{j:[n\cdot]}))) + \sum_{i=1}^{r} (X^{(n)}_{[n\cdot]+1-i:[n\cdot]}$$

$$- E(X^{(n)}_{[n\cdot]+1-i:[n\cdot]} 1_{(-\beta,\beta)}(X^{(n)}_{[n\cdot]+1-i:[n\cdot]}))), Z'_{1,\cdot} + \tilde{Z}_{3,\cdot} \Bigg) \geqslant \varepsilon_1 \Bigg) \leqslant \varepsilon_2 \tag{4.19}$$

holds for all $n \geqslant n_0$.

The proof of Theorem 4.2 requires some preparations. A formal summation of the limit variables of the lower order statistic (3.12) yield the centered series

$$\sum_{j=1}^{\infty}\left(\psi_1(S'_{\tau'_t(j)}) - E(\psi_1(S'_{\tau'_t(j)})1_{(-\beta,0)}(\psi_1(S'_{\tau'_t(j)})))\right). \tag{4.20}$$

The next Lemma proves that this series is indeed convergent in $D[0,1]$.

Lemma 4.2. *Under condition* (4.17) *series* (4.20) *converges in probability in* $(D[0,1],\|\cdot\|)$ *and it coincides almost surely with* $Z'_{1,\cdot}$ *given by* (4.6).

Proof. Notice that similar to (3.29) we have

$$\sum_{k=1}^{n}\psi_1(S'_{\tau'_t(k)}) = \sum_{j=1}^{\tau'^{(n)}_t}\psi_1(S'_j)1_{(0,t)}(V'_j). \tag{4.21}$$

Without restriction we may now assume that $\eta(-\infty,-\beta]=0$ and $\psi_1 > -\beta$ hold, cf. Janssen (1994, Section 4) for related truncation methods.

Under condition (4.8) one obtains by (4.10) of Janssen (1994)

$$\sum_{j=1}^{\infty}E(\psi_1(S'_j)) > -\infty \tag{4.22}$$

and elementary calculations for inverse binomial distributions prove

$$\sum_{j=1}^{\infty}E\left(\psi_1\left(S'_{\tau'_t(j)}\right)\right) = t\sum_{j=1}^{\infty}E(\psi_1(S_j)) \tag{4.23}$$

for each $t>0$. Notice that according to our assumption the indicator $1_{(-\beta,0)}(\cdot)$ disappears. In this special case (4.21) and (4.23) imply the result.

The general case can be treated by the following arguments. Kallenberg's result of (1974) shows that

$$t \mapsto \xi_n(t) := \sum_{j=1}^{\tau'^{(n)}_t}\psi_1(S'_j)1_{(0,t)}(V'_j) - t\sum_{j=1}^{\tau'^{(n)}_t}b_j \tag{4.24}$$

is convergent in $(D[0,1],\|\cdot\|)$ for $b_j := E(\psi_1(S_j))$ which abbreviates the centering coefficients. Here (4.21) and $\tau'^{(n)}_t \geqslant n$ should be taken into account. An inspectation of the means of (4.21) implies

$$\sum_{j=1}^{n}E\left(\psi_1(S'_{\tau'_t(j)})\right) = E\left(t\sum_{j=1}^{\tau'^{(n)}_t}b_j\right) \tag{4.25}$$

if first conditional expectations under $(V'_j)_{j\in\mathbb{N}}$ are considered.

Let $W_n(t)$ denote the nth partial sum of (4.20). Then we are now in the position to prove the convergence of

$$\|W_n(\cdot) - Z'_{1,\cdot}\| \to 0 \tag{4.26}$$

in probability. Since $W_n(0)=0$ we have

$$\|W_n(\cdot) - Z'_{1,\cdot}\| \leqslant \sup_{0\leqslant t\leqslant 1/n}|Z'_{1,t}| + \sup_{1/n\leqslant t\leqslant 1}|W_n(t) - Z'_{1,t}| \tag{4.27}$$

for each n. It is easy to see that

$$\sup_{0\leqslant t\leqslant s} |Z'_{1,t}| \to 0 \tag{4.28}$$

in probability as $s \downarrow 0$. (Recall that martingale arguments can be applied since $\mathrm{Var}(Z'_{1,s}) \to 0$ holds as $s \downarrow 0$.) On the other hand, a comparison of the partial sums (4.21), (4.24) and (4.25) yields

$$\sup_{1/n\leqslant t\leqslant 1} |W_n(t) - Z'_{1,t}| \leqslant \sup_{1/n\leqslant t\leqslant 1} |\xi_n(t) - Z'_{1,t}| + \sup_{1/n\leqslant t\leqslant 1} \left| t\left(\sum_{j=1}^{\tau_t'^{(n)}} b_j - E\left(\sum_{j=1}^{\tau_t'^{(n)}} b_j\right)\right)\right|. \tag{4.29}$$

Due to our assumptions the sequence $b_n \uparrow 0$ is increasing. Notice that

$$\sum_{n=1}^{\infty} |b_n|^{2-\alpha} \leqslant \sum_{n=1}^{\infty} E(|\psi_1(S'_n)|^{2-\alpha}) < \infty \tag{4.30}$$

holds. In view of Janssen (1994), (4.13) and condition (4.17) our sum (4.30) is finite. (Turn to a Lévy measure with inverse $y \mapsto -|\psi_1(y)|^{2-\alpha}$.) The subsequent Lemma 4.3 together with (4.24) implies that (4.29) converges to zero in probability. Thus our formulas (4.27) and (4.28) imply the result. □

Lemma 4.3. *Let $a_j \downarrow 0$ be a sequence of real numbers with $\sum_{j=1}^{\infty} a_j^{2-\alpha} < \infty$ for some $0 < \alpha < 1$. Define $Y_n(t) = \sum_{j=1}^{\tau_t'^{(n)}} a_j$. Then we have*

$$\sup_{1/n\leqslant t\leqslant 1} |Y_n(t) - E(Y_n(t))| \to 0 \tag{4.31}$$

in probability as $n \to \infty$.

Proof. It is easy to see that

$$s \mapsto \tau'^{(n)}_{1-s}$$

is a process with independent increments and $\tau_1^{(n)} = n$. Thus $s \mapsto Y_n(1-s) - E(Y_n(1-s))$ is a martingale. Now upper bounds for the variance of $Y_n(t)$ will be deduced. For these reasons let us first prove that

$$\mathrm{Var}(Y_n(t)) \leqslant a_n^2 \,\mathrm{Var}(\tau_t'^{(n)}) \tag{4.32}$$

holds. Since $\tau_t'^{(n)} \geqslant n$ we may assume for a moment that $a_1 = a_2 = \cdots = a_{n-1} = 0$ is satisfied. Then we may decompose

$$a_n \tau_t'^{(n)} = Y_n(t) + \sum_{j=n}^{\tau_t'^{(n)}} (a_n - a_j) =: Y_n(t) + Z_n(t).$$

By Hájek's Lemma 3.1, see Hájek (1968), we have $\mathrm{Cov}(Y_n(t), Z_n(t)) \geqslant 0$ and thus

$$\mathrm{Var}(Y_n(t)) \leqslant \mathrm{Var}(a_n \tau_t'^{(n)}) \tag{4.33}$$

implies statement (4.32). Elementary properties of geometric distributions imply

$$a_n^2 \operatorname{Var}(\tau_t'^{(n)}) \leqslant n a_n^2 \operatorname{Var}(\tau_t'^{(1)}) \leqslant \frac{n a_n^2}{t^2} \tag{4.34}$$

for each $t > 0$. This inequality can be combined with the Birnbaum Marshall inequalities for martingales, see Shorack and Wellner (1986, p. 873). Define $v(s) = \operatorname{Var}(Y_n(1-s))$. Then

$$P\left(\sup_{0 \leqslant s \leqslant 1-1/n} (1-s)|Y_n(1-s)| \geqslant \varepsilon\right) \leqslant \varepsilon^{-2} \int_{[0,1-1/n]} (1-s)^2 \, \mathrm{d}v(s) \tag{4.35}$$

holds. Integration by parts implies

$$\int_{[0,1-1/n]} (1-s)^2 \, \mathrm{d}v = n^{-2} v(1-1/n) + 2 \int_{[0,1-1/n]} (1-s) v(s) \, \mathrm{d}s. \tag{4.36}$$

Applying (4.32) and (4.34) we obtain the following upper bound of (4.36), namely

$$n a_n^2 \left(1 + 2 \int_{[0,1-1/n]} (1-s)^{-1} \, \mathrm{d}s\right) = n a_n^2 (1 + 2 \log n). \tag{4.37}$$

By our assumptions we have that $n a_n^{2-\alpha}$ is bounded and (4.35) converges to zero as $n \to \infty$. $\square$

The proof of Theorem 4.2.

The present proof will be prepared by convergence results about functions in $D[0,1]$. Although the Skorohod space $(D[0,1], d)$ is no topological group the following assertion holds. Under conditions (4.38) the convergence of $x_n \to y$ and $y_n \to y$ in $D[0,1]$ implies convergence of the sum $x_n + y_n \to x + y$ with respect to d (Whitt, 1980).

$$\begin{aligned} &x, y \text{ have no joint discontinuity points and they have} \\ &\text{no jumps in the upper end point } t = 1. \end{aligned} \tag{4.38}$$

Since this fact may be known we will only sketch the analytic proof along the lines of Billingsley (1968, Section 14) using his notation.

If I is an interval define

$$w_x(I) = \sup\{|x(s) - x(t)|: s, t \in I\}.$$

For each $\varepsilon > 0$ we may choose discontinuity points $0 < t_1 < t_2 < \cdots < t_{r-1} < 1$ of x with

$$w_x[t_{i-1}, t_i) < \varepsilon \quad \text{for} \quad 1 \leqslant i \leqslant r-1$$

and

$$w_x[t_{r-1}, 1] < \varepsilon,$$

where $t_0 = 0, t_r = 1$. Observe that for each $\delta > 0$ we have

$$w_{x-x_n}[t_{i-1} + \delta, t_i - \delta] \leqslant 2\varepsilon$$

and

$$w_{x-x_n}[t_{r-1} + \delta, 1] \leqslant 2\varepsilon$$

for n large enough.

The same assertion holds for discontinuity points $s_1 < s_2 < \cdot < s_{q-1}$ of y which are different from the t's.

Choose now $\delta \leqslant \varepsilon$ small enough such that $|t_i - s_j| > 8\delta$ holds for all pairs (i,j) and

$$A := \bigcup_{i=1}^{r-1} (t_i - 4\delta, t_i + 4\delta) \cup \bigcup_{j=1}^{q-1} (s_j - 4\delta, s_j + 4\delta)$$

is a union of pairwise disjoint intervals. Let now $\lambda_n : [0,1] \to [0,1]$ be strictly increasing and continuous with $\lambda_n(0) = 0$, $\lambda_n(1) = 1$ such that $\|x_n \circ \lambda_n - x\| < \delta$ and $\sup_{0 \leqslant t \leqslant 1} |\lambda_n(t) - t| < \delta$ holds for large n and let ρ_n be its counterpart for the y's, namely

$$\|y_n \circ \rho_n - y\| < \delta \quad \text{and} \quad \sup_{0 \leqslant t \leqslant 1} |\rho_n(t) - t| < \delta.$$

Define now a new continuous function $\lambda_n^*(t) = t$ for t outside of A and

$$\lambda_n^* = \lambda_n \quad \text{on } [t_i - 2\delta, t_i + 2\delta],$$

$$\lambda_n^* = \rho_n \quad \text{on } [s_j - 2\delta, s_j + 2\delta].$$

On the remaining intervals of the type $[t_i - 4\delta, t_i - 2\delta]$ the function λ_n^* is assumed to be linear and continuous in the endpoint. If we put everything together we have

$$\|(x_n + y_n) \circ \lambda_n^* - (x + y)\| \leqslant 4\varepsilon.$$

Thus (b) is proved.

If we now put all technical details together our Theorem 4.2 can be proved. We will restrict ourselves to (b) since the proof of (a) is similar. As consequence of Lemma 4.2 we may choose positive integers k and r with

$$P\left(\left\| \sum_{j=1}^{k} \left(\psi_1(S'_{\tau'^{(j)}_{\cdot}}) - E(\psi_1(S'_{\tau'^{(j)}_{\cdot}}) 1_{(-\beta,0)}(\psi_1(S'_{\tau'^{(j)}_{\cdot}}))) \right) \right.\right.$$
$$+ \sum_{i=1}^{r} \left(\psi_2(\tilde{S}_{\tilde{\tau}^{(i)}_{\cdot}}) - E(\psi_2(\tilde{S}_{\tilde{\tau}^{(i)}_{\cdot}}) 1_{(0,\beta)}(\psi_2(\tilde{S}_{\tilde{\tau}^{(i)}_{\cdot}}))) \right)$$
$$\left.\left. - (Z'_{1,\cdot} + \tilde{Z}_{3,\cdot}) \right\| \geqslant \varepsilon_1/2 \right) \leqslant \varepsilon_2/2. \tag{4.39}$$

From now on k and r are fixed. It is easy to see that Theorem 3.1 and (3.10) imply almost sure convergence of

$$\sum_{j=1}^{k} X^{(n)}_{j:[n\cdot]} \to \sum_{j=1}^{k} \psi_1(S'_{\tau'^{(j)}_{\cdot}}) \tag{4.40}$$

and

$$\sum_{i=1}^{r} X_{[n\cdot]+1-i:[n\cdot]} \to \sum_{i=1}^{r} \psi_2(\tilde{S}_{\tilde{\tau}^{(j)}_{\cdot}}) \tag{4.41}$$

both in $D[0,1]$ since the left-hand side of (4.40) decreases for $t \geqslant k/n$. Since $V'_1, V'_2, \ldots,$ $\tilde{V}_1, \tilde{V}_2, \ldots$ are all different with probability one we see that the limit processes (4.40) and (4.41) have no joint jumps almost surely and the process does not jump at $t = 1$ again with probability one. According to statement (4.38) the sum

$$\sum_{j=1}^{k} X_{j:[n\cdot]}^{(n)} + \sum_{i=1}^{r} X_{[n\cdot]+1-i:[n\cdot]}^{(n)} \tag{4.42}$$

is almost surely convergent in $(D[0,1], d)$. On the other hand, the dominated convergence theorem proves pointwise convergence of

$$\sum_{j=1}^{k} E\left(X_{j:[nt]}^{(n)} 1_{(-\beta,\beta)}(X_{j:[nt]}^{(n)}) \right) \to \sum_{j=1}^{k} E\left(S'_{\tau_t'^{(j)}} 1_{(-\beta,0)}(S'_{\tau_t'^{(j)}}) \right). \tag{4.43}$$

Notice that the right-hand side is continuous in t. The uniform convergence of (4.43) in t can be proved as follows. If we substitute the truncation function $x \mapsto x1_{(-\beta,\beta)}(x)$ of (4.43) by the monotone function

$$x \mapsto \varphi(x) = -\beta 1_{(-\infty,-\beta)}(x) + x 1_{(-\beta,\beta)}(x) + \beta 1_{[\beta,\infty)}(x)$$

we get by the monotony uniform convergence of

$$E\left(\varphi(X_{j:[n\cdot]}^{(n)}) \right)$$

for fixed j since the limit is continuous in t. The same argument applies to the truncation function $-\beta 1_{(-\infty,-\beta]}$. In addition (3.10) implies that the expectations with respect to truncation functions $1_{(\beta,\infty)}$ converge uniformly to zero. Thus (4.43) is uniformly convergent and a similar result holds for the expectations of the upper order statistics. The combination of (4.40)–(4.42) together with their truncated expectations implies almost sure convergence of the properly centered sums (4.42) to the related sum of partial sums of $Z'_{1,\cdot}$ and $\tilde{Z}_{3,\cdot}$ in $(D[0,1], d)$. The desired assertion now follows from (4.39). $\square$

Finally, we will return to the stable process (1.4) of Example 1.1. The application of Theorem 4.1 requires the following lemma which implies the almost sure convergence result (1.8).

Lemma 4.4. (a) *The* $\alpha = 1/2$ *stable process* (1.4) *is equal in distribution to*

$$Z_t \overset{\mathcal{D}}{=} \frac{2}{\pi} \sum_{j=1}^{\infty} S_j^{-2} 1_{(0,t)}(V_j), \quad 0 < t \leqslant 1. \tag{4.44}$$

(b) *Let* F_t *denote the distribution function of* Z_t. *Then* F_t *has the inverse distribution function* $t^2 F_1^{-1}(\cdot)$ *and we may choose* $X_{n,i}$ *of* (1.6) *as*

$$X_{n,i} := n^{-2} F_1^{-1}(1 - \bar{U}_i), \quad 1 \leqslant i \leqslant n, \tag{4.45}$$

where $(\bar{U}_1, \ldots, \bar{U}_n)$ *is taken from* (2.4). *For this special representation we have*

$$\sum_{i=1}^{[nt]} X_{[nt]+1-i:[nt]}^{(n)} \to \frac{2}{\pi} \sum_{j=1}^{\infty} S_j^{-2} 1_{(0,t)}(V_j) \tag{4.46}$$

almost surely for each $t > 0$.

As consequence the almost sure invariance principle (1.8) *holds for the inter arrival times* (1.6).

Proof. (a) The stability of the process implies $Z_t \stackrel{\mathscr{D}}{=} t^2 Z_1$ and $F_t(x) = F_1(xt^{-2})$. Since $(1 - F_1(x))x^{1/2} \to 2/\sqrt{2\pi}$ holds for $x \to \infty$, see Feller (1971, p. 52, 64), we have

$$n(1 - F_{1/n}(x)) \to \eta([x, \infty)) = \frac{2}{\sqrt{2\pi}} x^{-1/2} \quad \text{for } x > 0.$$

Consequently, the inverse (3.9) for η is just $\psi_2(y) = 2/\pi y^{-2}$ for $y > 0$ and (4.44) holds.

(b) For the inverse of F_1 we obtain

$$u^2 F_1^{-1}(1 - u) \to 2/\pi \quad \text{as } u \to \infty.$$

The arguments used in the proof of Theorem 4.1 show that for each fixed k and fixed $t > 0$ we have almost sure convergence of

$$\sum_{i=1}^{k} n^{-2} F_1^{-1}(1 - \bar{U}_{i:[nt]}) \to \frac{2}{\pi} \sum_{j=1}^{k} S_{\tau_t^{(j)}}^{-2}, \tag{4.47}$$

where $\tau_t^{(j)}$ is defined by (2.3). Notice that the almost sure limit of the right-hand side of (4.47) is just (4.44), confer (3.29) and (4.9).

In the next step, an upper bound for the remaining sum of the lower $n - k$ order statistics will be derived. The concrete expression of the F_1-density implies

$$1 - F_1(x) \leqslant \eta([x, \infty))$$

and

$$0 \leqslant F_1^{-1}(1 - u) \leqslant \psi_2(u)$$

for each $0 < u < 1$. Thus

$$\begin{aligned} \sum_{i=k+1}^{[nt]} n^{-2} F_1^{-1}(1 - \bar{U}_{i:[nt]}^{(n)}) &\leqslant \sum_{i=k+1}^{n} n^{-2} F_1^{-1}(1 - \bar{U}_{i:n}^{(n)}) \\ &\leqslant \frac{2}{\pi} \left(\frac{S_{n+1}}{n} \right)^2 \sum_{i=k+1}^{\infty} S_i^{-2} \end{aligned}$$

holds. Notice that the right-hand side converges to zero almost surely if $\min(n, k) \to \infty$. This result combined with (4.47) implies the statement of Lemma 4.1(b). □

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